Hand in no. 1, 2, 3, 8 and 9 by November 6.

# Assignment 8

1. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Show that
$$|Ax| \leq \sqrt{\sum_{i,j} a_{ij}^2}\ |x|.$$

2. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be $C^2$ and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists some $\rho > 0$ such that
$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$
is a contraction. This provides a justification for Newton's method in finding roots for an equation.

4. Consider the iteration
$$x_{n+1} = \alpha x_n(1 - x_n),\ x_0 \in [0, 1]\ .$$
Find
   (a) The range of $\alpha$ so that $\{x_n\}$ remains in $[0, 1]$ .
   (b) The range of $\alpha$ so that the iteration has a unique fixed point 0 in $[0, 1]$.
   (c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.

5. Show that every continuous function from $[0, 1]$ to itself admits a fixed point. Here we don't need it a contraction. Suggestion: Consider the sign of $g(x) = f(x) - x$ at $0, 1$ where $f$ is the given function.

6. Let $f$ be continuously differentiable on $[a, b]$. Show that it has a differentiable inverse if and only if its derivative is not equal to 0 at every point. This is 2060 stuff.

7. Consider the function
$$f(x) = \frac{1}{2}x + x^2 \sin\frac{1}{x}, \quad x \neq 0,$$
and set $f(0) = 0$. Show that $f$ is differentiable at 0 with $f'(0) = 1/2$ but it has no local inverse at 0. Does it contradict the Inverse Function Theorem?

8. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x, y) = x - x^2,\ g(x, y) = y + xy$ . Show that it has a local inverse at $(0, 0)$. And then write down the inverse map so that its domain can be described explicitly.

9. Let $F$ be a continuously differentiable map from the open $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ whose Jacobian determinant is non-vanishing everywhere. Prove that it maps every open set in $U$ to an open set, that is, $F$ is an open map. Does its inverse $F^{-1} : F(U) \to U$ always exist?

10. Consider the function

$$h(x,y) = (x - y^2)(x - 3y^2), \quad (x,y) \in \mathbb{R}^2.$$

Show that the set $\{(x,y) : \ h(x,y) = 0\}$ cannot be expressed as a local graph of a $C^1$-function over the $x$ or $y$-axis near the origin. Explain why the Implicit Function Theorem is not applicable.